

STABILITY OF DIRECT IMAGES UNDER FROBENIUS MORPHISM

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ABSTRACT. Let X be a smooth projective variety over an algebraically field k with $\text{char}(k) = p > 0$ and $F : X \rightarrow X_1$ be the relative Frobenius morphism. When $\dim(X) = 1$, we prove that F_*W is a stable bundle for any stable bundle W (Theorem 2.3). As a step to study the question for higher dimensional X , we generalize the canonical filtration (defined by Joshi-Ramanan-Xia-Yu for curves) to higher dimensional X (Theorem 3.6).

1. INTRODUCTION

Let X be a smooth projective variety over an algebraically field k with $\text{char}(k) = p > 0$ and $F : X \rightarrow X_1$ be the relative Frobenius morphism. When $\dim(X) = 1$, Lange and Pauly proved that $F_*\mathcal{L}$ is a stable bundle for a line bundle \mathcal{L} (cf. [3, Proposition 1.2]). The first result in this paper is that stability of W implies stability of F_*W .

Recall that for a Galois étale G -cover $f : Y \rightarrow X$ and a semistable bundle W on Y , to prove semistability of f_*W , one uses the fact that $f^*(f_*W)$ decomposes into pieces of W^σ ($\sigma \in G$). To imitate this idea for $F : X \rightarrow X_1$, we need a similar decomposition of $V = F^*(F_*W)$. Indeed, use the canonical connection $\nabla : V \rightarrow V \otimes \Omega_X^1$, Joshi-Ramanan-Xia-Yu have defined in [1] for $\dim(X) = 1$ a canonical filtration

$$0 = V_p \subset V_{p-1} \subset \cdots \subset V_i \subset V_{i-1} \subset \cdots \subset V_1 \subset V_0 = V$$

such that $V_i/V_{i+1} \cong W \otimes (\Omega_X^1)^{\otimes i}$. For any $0 \neq \mathcal{E} \subset F_*W$, let

$$0 \subset V_m \cap F^*\mathcal{E} \subset \cdots \subset V_1 \cap F^*\mathcal{E} \subset V_0 \cap F^*\mathcal{E} = F^*\mathcal{E}$$

be the induced filtration. Then we can show (cf. Lemma 2.2)

$$\frac{V_{i-1} \cap F^*\mathcal{E}}{V_i \cap F^*\mathcal{E}} \neq 0 \quad \text{for} \quad 1 \leq i \leq m+1.$$

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Using the induced filtration and stability of $W \otimes (\Omega_X^1)^{\otimes i}$, we have

$$\mu(F_*W) - \mu(\mathcal{E}) \geq \frac{g-1}{p} \left(p-1 - \frac{2}{\text{rk}(\mathcal{E})} \cdot \sum_{i=1}^{m+1} (i-1) \text{rk}\left(\frac{V_{i-1} \cap F^*\mathcal{E}}{V_i \cap F^*\mathcal{E}}\right) \right).$$

When W is a line bundle, all $\frac{V_{i-1} \cap F^*\mathcal{E}}{V_i \cap F^*\mathcal{E}}$ must be line bundles and $\text{rk}(\mathcal{E}) = m+1$. Then above inequality implies the stability of F_*W immediately. For higher rank bundles W , we need more analysis of the rank of $\frac{V_{i-1} \cap F^*\mathcal{E}}{V_i \cap F^*\mathcal{E}}$.

It is a natural question to study F_*W for $\dim(X) = n > 1$. As the first step, we generalize the canonical filtration to higher dimensional X . Its definition can be generalized straightforwardly by using the canonical connection $\nabla : V \rightarrow V \otimes \Omega_X^1$. The second result of this paper is that there exists a canonical filtration

$$0 = V_{n(p-1)+1} \subset V_{n(p-1)} \subset \cdots \subset V_1 \subset V_0 = V = F^*(F_*W)$$

such that ∇ induces $V_i/V_{i+1} \cong W \otimes (\Omega_X^1)^{[i]}$, where $(\Omega_X^1)^{[i]} \subset (\Omega_X^1)^{\otimes i}$ is a subbundle given by a representation of $\text{GL}(n)$ (cf. Definition 3.4). In characteristic zero, $(\Omega_X^1)^{[i]} = \text{Sym}^i(\Omega_X^1)$. In characteristic $p > 0$, we have $(\Omega_X^1)^{[i]} \cong \text{Sym}^i(\Omega_X^1)$ for $i < p$. The general question would be: how to bound the instability of F_*W by instability of $W \otimes (\Omega_X^1)^{[i]}$?

When I was preparing the last section of this paper, Mehta and Pauly posted a preprint [4], in which they prove, in a different method, that semistability of W implies semistability of F_*W . But they do not prove that stability of W implies stability of F_*W .

2. THE CASE OF CURVES

Let k be an algebraically closed field of characteristic $p > 0$ and X be a smooth projective curve over k . Let $F : X \rightarrow X_1$ be the relative k -linear Frobenius morphism, where $X_1 := X \times_k k$ is the base change of X/k under the Frobenius $\text{Spec}(k) \rightarrow \text{Spec}(k)$. Let W be a vector bundle on X and $V = F^*(F_*W)$. It is known ([2, Theorem 5.1]) that V has an p -curvature zero connection $\nabla : V \rightarrow V \otimes \Omega_X^1$. In [1, Section 5], the authors defined a canonical filtration

$$(2.1) \quad 0 = V_p \subset V_{p-1} \subset \cdots \subset V_i \subset V_{i-1} \subset \cdots \subset V_1 \subset V_0 = V$$

where $V_1 = \ker(V = F^*F_*W \rightarrow W)$ and

$$(2.2) \quad V_{i+1} = \ker(V_i \xrightarrow{\nabla} V \otimes \Omega_X^1 \rightarrow V/V_i \otimes \Omega_X^1).$$

The following lemma belongs to them (cf. [1, Theorem 5.3]).

Lemma 2.1. (i) $V_0/V_1 \cong W$, $\nabla(V_{i+1}) \subset V_i \otimes \Omega_X^1$ for $i \geq 1$.

- (ii) $V_i/V_{i+1} \xrightarrow{\nabla} (V_{i-1}/V_i) \otimes \Omega_X^1$ is an isomorphism for $1 \leq i \leq p-1$.
- (iii) If $g \geq 2$ and W is semistable, then the canonical filtration (2.1) is nothing but the Harder-Narasimhan filtration.

Proof. (i) follows by the definition, which and (ii) imply (iii). To prove (ii), let $I_0 = F^*F_*\mathcal{O}_X$, $I_1 = \ker(F^*F_*\mathcal{O}_X \rightarrow \mathcal{O}_X)$ and

$$(2.3) \quad I_{i+1} = \ker(I_i \xrightarrow{\nabla} I_0 \otimes \Omega_X^1 \rightarrow I_0/I_i \otimes \Omega_X^1)$$

which is the canonical filtration (2.1) in the case $W = \mathcal{O}_X$.

(ii) is clearly a local problem, we can assume $X = \operatorname{Spec}(k[[x]])$ and $W = k[[x]]^{\oplus r}$. Then $V_0 := V = F^*(F_*W) = I_0^{\oplus r}$, $V_i = I_i^{\oplus r}$ and

$$(2.4) \quad V_i/V_{i+1} = (I_i/I_{i+1})^{\oplus r} \xrightarrow{\oplus \nabla} (I_{i-1}/I_i \otimes \Omega_X^1)^{\oplus r} = V_{i-1}/V_i \otimes \Omega_X^1.$$

Thus it is enough to show that

$$(2.5) \quad I_i/I_{i+1} \xrightarrow{\nabla} (I_{i-1}/I_i) \otimes \Omega_X^1$$

is an isomorphism. Locally, $I_0 = k[[x]] \otimes_{k[[x^p]]} k[[x]]$ and

$$(2.6) \quad \nabla : k[[x]] \otimes_{k[[x^p]]} k[[x]] \rightarrow I_0 \otimes_{\mathcal{O}_X} \Omega_X^1,$$

where $\nabla(g \otimes f) = g \otimes f' \otimes dx$. The \mathcal{O}_X -module

$$(2.7) \quad I_1 := \ker(k[[x]] \otimes_{k[[x^p]]} k[[x]] \rightarrow k[[x]])$$

has a basis $\{x^k \otimes 1 - 1 \otimes x^k\}_{1 \leq k \leq p-1}$. Notice that I_1 is also an ideal of the \mathcal{O}_X -algebra $I_0 = k[[x]] \otimes_{k[[x^p]]} k[[x]]$, let $\alpha = x \otimes 1 - 1 \otimes x$, then $\alpha^k \in I_1$. It is easy to see that $\alpha, \alpha^2, \dots, \alpha^{p-1}$ is a basis of the \mathcal{O}_X -module I_1 (notice that $\alpha^p = x^p \otimes 1 - 1 \otimes x^p = 0$), and

$$(2.8) \quad \nabla(\alpha^k) = -k\alpha^{k-1} \otimes dx.$$

Thus, as a free \mathcal{O}_X -module, I_i has a basis $\{\alpha^i, \alpha^{i+1}, \dots, \alpha^{p-1}\}$, which means that I_i/I_{i+1} has a basis α^i , $(I_{i-1}/I_i) \otimes \Omega_X^1$ has a basis $\alpha^{i-1} \otimes dx$ and $\nabla(\alpha^i) = -i\alpha^{i-1} \otimes dx$. Therefore ∇ induces the isomorphism (2.5) since $(i, p) = 1$, which implies the isomorphism in (ii). \square

Lemma 2.2. *Let $\mathcal{E} \subset F_*W$ be a nontrivial subsheaf and let*

$$(2.9) \quad 0 \subset V_m \cap F^*\mathcal{E} \subset \dots \subset V_1 \cap F^*\mathcal{E} \subset V_0 \cap F^*\mathcal{E} = F^*\mathcal{E}$$

be the induced filtration. Then

$$(2.10) \quad \frac{V_{i-1} \cap F^*\mathcal{E}}{V_i \cap F^*\mathcal{E}} \neq 0 \quad \text{for} \quad 1 \leq i \leq m+1.$$

Proof. Firstly, by adjunction formula, $F^*\mathcal{E} \hookrightarrow V = F^*(F_*W) \twoheadrightarrow W$ is nontrivial. Thus $V_0 \cap F^*\mathcal{E}/V_1 \cap F^*\mathcal{E}$ is nontrivial. On the other hand, for any $i \geq 2$, the morphism $V_{i-1} \cap F^*\mathcal{E} \hookrightarrow V = F^*(F_*W) \twoheadrightarrow W$ is trivial, which implies, by adjunction formula, that there is no subsheaf $j : \mathcal{E}' \hookrightarrow F_*W$ such that $F^*j : V_{i-1} \cap F^*\mathcal{E} \cong F^*\mathcal{E}' \hookrightarrow V$ is the inclusion $V_{i-1} \cap F^*\mathcal{E} \hookrightarrow V$. However, by the definition of canonical filtration (2.1), $V_{i-1} \cap F^*\mathcal{E} = V_i \cap F^*\mathcal{E}$ implies that

$$(2.11) \quad \nabla(V_{i-1} \cap F^*\mathcal{E}) \subset (V_{i-1} \cap F^*\mathcal{E}) \otimes \Omega_X^1.$$

By [2, Theorem 5.1], this means that there is an $j : \mathcal{E}' \hookrightarrow F_*W$ such that $F^*j : V_{i-1} \cap F^*\mathcal{E} \cong F^*\mathcal{E}' \hookrightarrow V$ is the inclusion $V_{i-1} \cap F^*\mathcal{E} \hookrightarrow V$. We get contradiction. \square

Theorem 2.3. *If W is a stable vector bundle, then F_*W is a stable vector bundle. In particular, if W is semistable, then F_*W is semistable.*

Proof. Let $\mathcal{E} \subset F_*W$ be a nontrivial subbundle and

$$(2.12) \quad 0 \subset V_m \cap F^*\mathcal{E} \subset \cdots \subset V_1 \cap F^*\mathcal{E} \subset V_0 \cap F^*\mathcal{E} = F^*\mathcal{E}$$

be the induced filtration. Let $r_{i-1} = \text{rk}(\frac{V_{i-1} \cap F^*\mathcal{E}}{V_i \cap F^*\mathcal{E}})$ be the ranks of quotients. Then, by the filtration (2.12), we have

$$(2.13) \quad \mu(F^*\mathcal{E}) = \frac{1}{\text{rk}(F^*\mathcal{E})} \sum_{i=1}^{m+1} r_{i-1} \mu(\frac{V_{i-1} \cap F^*\mathcal{E}}{V_i \cap F^*\mathcal{E}}).$$

By Lemma 2.1, $V_{i-1}/V_i \cong W \otimes (\Omega_X^1)^{\otimes(i-1)}$ is stable, we have

$$(2.14) \quad \mu(\frac{V_{i-1} \cap F^*\mathcal{E}}{V_i \cap F^*\mathcal{E}}) \leq \mu(W) + 2(g-1)(i-1).$$

Then, notice that $\mu(V) = \mu(W) + (p-1)(g-1)$, we have

$$(2.15) \quad \mu(F_*W) - \mu(\mathcal{E}) \geq \frac{2g-2}{p \cdot \text{rk}(\mathcal{E})} \cdot \sum_{i=1}^{m+1} (\frac{p+1}{2} - i) r_{i-1}$$

which becomes into an equality if and only if the inequalities in (2.14) become into equalities.

It is clear by (2.15) that $\mu(F_*W) - \mu(\mathcal{E}) > 0$ if $m \leq \frac{p-1}{2}$. Thus we assume that $m > \frac{p-1}{2}$. On the other hand, since the isomorphisms $V_i/V_{i+1} \xrightarrow{\nabla} (V_{i-1}/V_i) \otimes \Omega_X^1$ in Lemma 2.1 (ii) induce the injections

$$(2.16) \quad \frac{V_i \cap F^*\mathcal{E}}{V_{i+1} \cap F^*\mathcal{E}} \hookrightarrow \frac{V_{i-1} \cap F^*\mathcal{E}}{V_i \cap F^*\mathcal{E}} \otimes \Omega_X^1$$

we have

$$(2.17) \quad r_0 \geq r_1 \geq \cdots \geq r_{i-1} \geq r_i \geq \cdots \geq r_m.$$

Then, when $m > \frac{p-1}{2}$, we can write

$$(2.18) \quad \sum_{i=1}^{m+1} \left(\frac{p+1}{2} - i \right) r_{i-1} = \sum_{i=1}^{\frac{p-1}{2}} i \cdot r_{\frac{p-1}{2}-i} - \sum_{i=1}^{m-\frac{p-1}{2}} i \cdot r_{\frac{p-1}{2}+i}$$

Note that $m \leq p-1$, use (2.17) and (2.18), we have

$$(2.19) \quad \sum_{i=1}^{m+1} \left(\frac{p+1}{2} - i \right) r_{i-1} \geq \sum_{i=1}^{m-\frac{p-1}{2}} i \cdot (r_{\frac{p-1}{2}-i} - r_{\frac{p-1}{2}+i}) \geq 0.$$

Thus we always have

$$(2.20) \quad \mu(F_*W) - \mu(\mathcal{E}) \geq \frac{2g-2}{p \cdot \text{rk}(\mathcal{E})} \cdot \sum_{i=1}^{m+1} \left(\frac{p+1}{2} - i \right) r_{i-1} \geq 0.$$

If $\mu(F_*W) - \mu(\mathcal{E}) = 0$, then (2.15) and (2.19) become into equalities. That (2.15) becomes into an equality implies inequalities in (2.14) become into equalities, which means $r_0 = r_1 = \cdots = r_m = \text{rk}(W)$. Then that (2.19) become into equalities implies $m = p-1$. Altogether imply $\mathcal{E} = F_*W$, we get contradiction. Hence F_*W is a stable vector bundle whenever W is stable. \square

3. GENERALIZATIONS TO HIGHER DIMENSION VARIETIES

Let X be a smooth projective variety over k of dimension n and $F : X \rightarrow X_1$ be the relative k -linear Frobenius morphism, where $X_1 := X \times_k k$ is the base change of X/k under the Frobenius $\text{Spec}(k) \rightarrow \text{Spec}(k)$. Let W be a vector bundle on X and $V = F^*(F_*W)$. We have the straightforward generalization of the canonical filtration to higher dimensional varieties.

Definition 3.1. Let $V_0 := V = F^*(F_*W)$, $V_1 = \ker(F^*(F_*W) \rightarrow W)$

$$(3.1) \quad V_{i+1} := \ker(V_i \xrightarrow{\nabla} V \otimes_{\mathcal{O}_X} \Omega_X^1 \rightarrow (V/V_i) \otimes_{\mathcal{O}_X} \Omega_X^1)$$

where $\nabla : V \rightarrow V \otimes_{\mathcal{O}_X} \Omega_X^1$ is the canonical connection (cf. [2, Theorem]).

We first consider the special case $W = \mathcal{O}_X$ and give some local descriptions. Let $I_0 = F^*(F_*\mathcal{O}_X)$, $I_1 = \ker(F^*F_*\mathcal{O}_X \rightarrow \mathcal{O}_X)$ and

$$(3.2) \quad I_{i+1} = \ker(I_i \xrightarrow{\nabla} I_0 \otimes_{\mathcal{O}_X} \Omega_X^1 \rightarrow I_0/I_i \otimes_{\mathcal{O}_X} \Omega_X^1).$$

Locally, let $X = \text{Spec}(A)$, $I_0 = A \otimes_{A^p} A$, where $A = k[[x_1, \dots, x_n]]$, $A^p = k[[x_1^p, \dots, x_n^p]]$. Then the canonical connection $\nabla : I_0 \rightarrow I_0 \otimes_{\mathcal{O}_X} \Omega_X^1$

is locally defined by

$$(3.3) \quad \nabla(g \otimes_{A^p} f) = \sum_{i=1}^n (g \otimes_{A^p} \frac{\partial f}{\partial x_i}) \otimes_A dx_i$$

Notice that I_0 has an A -algebra structure such that $I_0 = A \otimes_{A^p} A \twoheadrightarrow A$ is a homomorphism of A -algebras, its kernel I_1 contains elements

$$(3.4) \quad \alpha_1^{k_1} \alpha_2^{k_2} \cdots \alpha_n^{k_n}, \text{ where } \alpha_i = x_i \otimes_{A^p} 1 - 1 \otimes_{A^p} x_i, \sum_{i=1}^n k_i \geq 1.$$

Since $\alpha_i^p = x_i^p \otimes_{A^p} 1 - 1 \otimes_{A^p} x_i^p = 0$, the set $\{\alpha_1^{k_1} \cdots \alpha_n^{k_n} \mid k_1 + \cdots + k_n \geq 1\}$ has $p^n - 1$ elements. In fact, we have

Lemma 3.2. *Locally, as free A -modules, we have, for all $i \geq 1$,*

$$(3.5) \quad I_i = \bigoplus_{k_1 + \cdots + k_n \geq i} (\alpha_1^{k_1} \cdots \alpha_n^{k_n}) A.$$

Proof. We first prove for $i = 1$ that $\{\alpha_1^{k_1} \cdots \alpha_n^{k_n} \mid k_1 + \cdots + k_n \geq 1\}$ is a basis of I_1 locally. By definition, I_1 is locally free of rank $p^n - 1$, thus it is enough to show that as an A -module I_1 is generated locally by $\{\alpha_1^{k_1} \cdots \alpha_n^{k_n} \mid k_1 + \cdots + k_n \geq 1\}$ since it has exactly $p^n - 1$ elements.

It is easy to see that as an A -module I_1 is locally generated by $\{x_1^{k_1} \cdots x_n^{k_n} \otimes_{A^p} 1 - 1 \otimes_{A^p} x_1^{k_1} \cdots x_n^{k_n} \mid k_1 + \cdots + k_n \geq 1, 0 \leq k_i \leq p-1\}$. It is enough to show any $x_1^{k_1} \cdots x_n^{k_n} \otimes_{A^p} 1 - 1 \otimes_{A^p} x_1^{k_1} \cdots x_n^{k_n}$ is a linear combination of $\{\alpha_1^{k_1} \cdots \alpha_n^{k_n} \mid k_1 + \cdots + k_n \geq 1\}$. The claim is obvious when $k_1 + \cdots + k_n = 1$, we consider the case $k_1 + \cdots + k_n > 1$. Without loss generality, assume $k_n \geq 1$ and there are $f_{j_1, \dots, j_n} \in A$ such that

$$x_1^{k_1} \cdots x_n^{k_n-1} \otimes_{A^p} 1 - 1 \otimes_{A^p} x_1^{k_1} \cdots x_n^{k_n-1} = \sum_{j_1 + \cdots + j_n \geq 1} (\alpha_1^{j_1} \cdots \alpha_n^{j_n}) \cdot f_{j_1, \dots, j_n}.$$

Then we have

$$\begin{aligned} x_1^{k_1} \cdots x_n^{k_n} \otimes_{A^p} 1 - 1 \otimes_{A^p} x_1^{k_1} \cdots x_n^{k_n} &= \sum_{j_1 + \cdots + j_n \geq 1} (\alpha_1^{j_1} \cdots \alpha_n^{j_n+1}) \cdot f_{j_1, \dots, j_n} \\ &+ \sum_{j_1 + \cdots + j_n \geq 1} (\alpha_1^{j_1} \cdots \alpha_n^{j_n}) \cdot f_{j_1, \dots, j_n} x_n + \alpha_n \cdot (x_1^{k_1} \cdots x_n^{k_n-1}). \end{aligned}$$

For $i > 1$, to prove the lemma, we first show

$$(3.6) \quad \nabla(\alpha_1^{k_1} \cdots \alpha_n^{k_n}) = - \sum_{i=1}^n k_i (\alpha_1^{k_1} \cdots \alpha_i^{k_i-1} \cdots \alpha_n^{k_n}) \otimes_A dx_i$$

Indeed, (3.6) is true when $k_1 + \cdots + k_n = 1$. If $k_1 + \cdots + k_n > 1$, we assume $k_n \geq 1$ and $\alpha_1^{k_1} \cdots \alpha_n^{k_n-1} = \sum g_j \otimes_{A^p} f_j$. Then

$$\alpha_1^{k_1} \cdots \alpha_n^{k_n} = \sum_j x_n g_j \otimes_{A^p} f_j - \sum_j g_j \otimes_{A^p} f_j x_n.$$

Use (3.3), straightforward computations show

$$\nabla(\alpha_1^{k_1} \cdots \alpha_n^{k_n}) = \alpha_n \nabla(\alpha_1^{k_1} \cdots \alpha_n^{k_n-1}) - (\alpha_1^{k_1} \cdots \alpha_n^{k_n-1}) \otimes_A dx_n$$

which implies (3.6). Now we can assume the lemma is true for I_{i-1} and recall that $I_i = \ker(I_{i-1} \xrightarrow{\nabla} I_0 \otimes_A \Omega_X^1 \rightarrow (I_0/I_{i-1}) \otimes_A \Omega_X^1)$. For any

$$\beta = \sum_{k_1 + \cdots + k_n \geq i-1} (\alpha_1^{k_1} \cdots \alpha_n^{k_n}) \cdot f_{k_1, \dots, k_n} \in I_{i-1}, \quad f_{k_1, \dots, k_n} \in A,$$

by using (3.6), we see that $\beta \in I_i$ if and only if

$$(3.7) \quad \sum_{k_1 + \cdots + k_n = i-1} (\alpha_1^{k_1} \cdots \alpha_j^{k_j-1} \cdots \alpha_n^{k_n}) \cdot k_j f_{k_1, \dots, k_n} \in I_{i-1}$$

for all $1 \leq j \leq n$. Since $\{\alpha_1^{k_1} \cdots \alpha_n^{k_n} \mid k_1 + \cdots + k_n \geq 1\}$ is a basis of I_1 locally and the lemma is true for I_{i-1} , (3.7) is equivalent to

$$(3.8) \quad \text{For given } (k_1, \dots, k_n) \text{ with } k_1 + \cdots + k_n = i-1 \\ k_j f_{k_1, \dots, k_n} = 0 \text{ for all } j = 1, \dots, n$$

which implies $f_{k_1, \dots, k_n} = 0$ whenever $k_1 + \cdots + k_n = i-1$. Thus I_i is generated by $\{\alpha_1^{k_1} \cdots \alpha_n^{k_n} \mid k_1 + \cdots + k_n \geq i\}$. \square

Lemma 3.3. (i) $I_i = 0$ when $i > n(p-1)$, and $\nabla(I_{i+1}) \subset I_i \otimes \Omega_X^1$ for $i \geq 1$.

(ii) $I_i/I_{i+1} \xrightarrow{\nabla} (I_{i-1}/I_i) \otimes \Omega_X^1$ are injective in the category of vector bundles for $1 \leq i \leq n(p-1)$. In particular, their composition

$$(3.9) \quad \nabla^i : I_i/I_{i+1} \rightarrow (I_0/I_1) \otimes_{\mathcal{O}_X} (\Omega_X^1)^{\otimes i} = (\Omega_X^1)^{\otimes i}$$

is injective in the category of vector bundles.

Proof. (i) follows from Lemma 3.2 and Definition 3.1. (ii) follows from (3.6). \square

In order to describe the image of ∇^i in (3.9), we recall a $\mathrm{GL}(n)$ -representation $V^{[\ell]} \subset V^{\otimes \ell}$ where V is the standard representation of $\mathrm{GL}(n)$. Let S_ℓ be the symmetric group of ℓ elements with the action

on $V^{\otimes \ell}$ by $(v_1 \otimes \cdots \otimes v_\ell) \cdot \sigma = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}$ for $v_i \in V$ and $\sigma \in S_\ell$. Let e_1, \dots, e_n be a basis of V , for $k_i \geq 0$ with $k_1 + \cdots + k_n = \ell$ define

$$(3.10) \quad v(k_1, \dots, k_n) = \sum_{\sigma \in S_\ell} (e_1^{\otimes k_1} \otimes \cdots \otimes e_n^{\otimes k_n}) \cdot \sigma$$

Definition 3.4. Let $V^{[\ell]} \subset V^{\otimes \ell}$ be the linear subspace generated by all vectors $v(k_1, \dots, k_n)$ for all $k_i \geq 0$ satisfying $k_1 + \cdots + k_n = \ell$. It is clearly a sub-representation of $\mathrm{GL}(V)$. If \mathcal{V} is a vector bundle of rank n , the subbundle $\mathcal{V}^{[\ell]} \subset \mathcal{V}^{\otimes \ell}$ is defined to be the associated bundle of the frame bundle of \mathcal{V} (which is a principal $\mathrm{GL}(n)$ -bundle) through the representation $V^{[\ell]}$.

In characteristic zero, $V^{[\ell]}$ is nothing but $\mathrm{Sym}^\ell(V)$. When $\mathrm{char}(k) = p > 0$, we have $v(k_1, \dots, k_n) = 0$ if one of k_1, \dots, k_n is bigger than $p - 1$. Thus $V^{[\ell]}$ is in fact spanned by

$$(3.11) \quad \{v(k_1, \dots, k_n) \mid 0 \leq k_i \leq p - 1, k_1 + \cdots + k_n = \ell\}.$$

In general, $V^{[\ell]}$ is not isomorphic to $\mathrm{Sym}^\ell(V)$, but it is easy to see

$$(3.12) \quad V^{[\ell]} \cong \mathrm{Sym}^\ell(V) \quad \text{when} \quad 0 < \ell < p.$$

Lemma 3.5. *With the notation in Definition 3.4, the composition*

$$(3.13) \quad \nabla^\ell : I_\ell / I_{\ell+1} \rightarrow (\Omega_X^1)^{\otimes \ell}$$

of the \mathcal{O}_X -morphisms in Lemma 3.3 (ii) has image $(\Omega_X^1)^{[\ell]} \subset (\Omega_X^1)^{\otimes \ell}$.

Proof. It is enough to prove the lemma locally. By Lemma 3.2, $I_\ell / I_{\ell+1}$ is locally generated by

$$(3.14) \quad \{\alpha_1^{k_1} \cdots \alpha_n^{k_n} \mid k_1 + \cdots + k_n = \ell\}.$$

By using formula (3.6), we have

$$(3.15) \quad \nabla^\ell(\alpha_1^{k_1} \cdots \alpha_n^{k_n}) = (-1)^\ell \sum_{\sigma \in S_\ell} (dx_1^{\otimes k_1} \otimes \cdots \otimes dx_n^{\otimes k_n}) \cdot \sigma$$

which implies that $\nabla^\ell(I_\ell / I_{\ell+1}) = (\Omega_X^1)^{[\ell]} \subset (\Omega_X^1)^{\otimes \ell}$. □

Theorem 3.6. *The filtration defined in Definition 3.1 is*

$$(3.16) \quad 0 = V_{n(p-1)+1} \subset V_{n(p-1)} \subset \cdots \subset V_1 \subset V_0 = V = F^*(F_*W)$$

which has the following properties

- (i) $\nabla(V_{i+1}) \subset V_i \otimes \Omega_X^1$ for $i \geq 1$, and $V_0 / V_1 \cong W$.
- (ii) $V_i / V_{i+1} \xrightarrow{\nabla} (V_{i-1} / V_i) \otimes \Omega_X^1$ are injective morphisms of vector bundles for $1 \leq i \leq n(p-1)$, which induced isomorphisms

$$\nabla^i : V_i / V_{i+1} \cong W \otimes_{\mathcal{O}_X} (\Omega_X^1)^{[i]}, \quad 0 \leq i \leq n(p-1).$$

In particular, $V_i / V_{i+1} \cong W \otimes_{\mathcal{O}_X} \mathrm{Sym}^i(\Omega_X^1)$ for $i < p$.

Proof. It is a local problem to prove the theorem. Thus $V_{n(p-1)+1} = 0$ follows from Lemma 3.2, and (ii) follows from Lemma 3.3 and Lemma 3.5. (i) is nothing but the definition. \square

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